ELSEVIER

Journal of Geometry and Physics 17 (1995) 95–124

JOURNAL OF GEOMETRY AND PHYSICS

KdV type hierarchies, the string equation and $W_{1+\infty}$ constraints

Johan van de Leur [1]

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands
e-mail: vdleur@math.utwente.nl

Received 13 June 1994

Abstract

To every partition $n = n_1 + n_2 + \cdots + n_s$ one can associate a vertex operator realization of the Lie algebras a_∞ and $\widehat{gl}_n$. Using this construction we make reductions of the s-component KP hierarchy, reductions which are related to these partitions. In this way we obtain matrix KdV type equations. Now assuming that (1) τ is a τ-function of the $[n_1, n_2, \dots, n_s]$th reduced KP hierarchy and (2) τ satisfies a 'natural' string equation, we prove that τ also satisfies the vacuum constraints of the $W_{1+\infty}$ algebra.

Keywords: KdV hierarchies; Strings;
1991 MSC: 35 Q 53, 73 K 03

0. Introduction

In recent years KdV type hierarchies have been related to 2D gravity. To be slightly more precise (see [Dij] for the details and references), the square root of the partition function of the Hermitian $(n-1)$-matrix model in the continuum limit is the τ-function of the n-reduced Kadomtsev–Petviashvili (KP) hierarchy. Hence, the $(n-1)$-matrix model corresponds to nth Gelfand–Dickey hierarchy. For $n = 2, 3$ these hierarchies are better known as the KdV and Boussinesque hierarchy, respectively. The partition function is then characterized by the so-called string equation:

[1] Part of this research was done while the author was working at the Mathematical Institute of the University of Utrecht. In this period he was supported by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

Elsevier Science B.V.
SSDI 0393-0440(94)00039-5

$$L_{-1}\tau = \frac{1}{n}\frac{\partial \tau}{\partial x_1}, \tag{0.1}$$

where L_{-1} is an element of the $c = n$ Virasoro algebra, which is related to the principal realization of the affine lie algebra $\widehat{sl}_n$, or rather $\widehat{gl}_n$. Let $\alpha_k = -kx_{-k}, 0, \partial/\partial x_k$ for $k < 0, k = 0, k > 0$, respectively, then

$$L_k = \frac{1}{2n}\sum_{\ell\in\mathbb{Z}} :\alpha_{-\ell}\alpha_{\ell+nk}: + \delta_{0k}\frac{n^2-1}{24n}. \tag{0.2}$$

By making the shift $x_{n+1} \mapsto x_{n+1} + n/(n+1)$, we modify the origin of the τ-function and thus obtain the following form of the string equation:

$$L_{-1}\tau = 0. \tag{0.3}$$

Actually, it can be shown [FKN, G] that the above conditions, nth reduced KP and Eq. (0.3) (which from now on we will call the string equation), on a τ-function of the KP hierarchy imply more general constraints, viz. the vacuum constraints of the $W_{1+\infty}$ algebra. This last condition is reduced to the vacuum conditions of the W_n algebra when some redundant variables are eliminated.

The $W_{1+\infty}$ algebra is the central extension of the Lie algebra of differential operators on $\mathbb{C}^\times$. This central extension was discovered by Kac and Peterson in 1981 [KP3] (see also [Ra, KRa]). It has as basis the operators $W_k^{(\ell+1)} = -t^{k+\ell}(\partial/\partial t)^\ell$, $\ell \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, together with the central element c. There is a well-known way to express these elements in the elements of the Heisenberg algebra, the α_k's. The $W_{1+\infty}$ constraints then are

$$\hat{W}_k^{(\ell+1)}\tau = \{W_k^{(\ell+1)} + \delta_{k,0}c_\ell\}\tau = 0 \quad \text{for } \ell \geq 0, k \geq -\ell. \tag{0.4}$$

For the above τ-function, $\hat{W}_k^{(1)} = -\alpha_{nk}$ and $\hat{W}_k^{(2)} = L_k - [(nk+1)/n]\alpha_{nk}$.

It is well-known that the n-reduced KP hierarchy is related to the principal realization (a vertex realization) of the basic module of $\widehat{sl}_n$. However there are many inequivalent vertex realizations. Kac and Peterson [KP1] and independently Lepowsky [L] showed that for the basic representation of a simply-laced affine Lie algebra these different realizations are parametrized by the conjugacy classes of the Weyl group of the corresponding finite dimensional Lie algebra. Hence, for the case of $\widehat{sl}_n$ they are parametrized by the partitions $n = n_1 + n_2 + \cdots + n_s$ of n. An explicit description of these realizations was given in [TV] (see also Section 2). There the construction was given in such a way that it was possible to make reductions of the KP-hierarchy. In all these constructions a 'natural' Virasoro algebra played an important role. A natural question now is: If τ is a τ-function of this $[n_1, n_2, \ldots, n_s]$th reduced KP hierarchy and τ satisfies the string equation (0.3), where L_{-1} is an element of this new Virasoro algebra, does τ also satisfy some corresponding $W_{1+\infty}$ constraints? In this paper we give a positive answer to this question. As will be shown in Section 6, there exists a 'natural' $W_{1+\infty}$ algebra for which (0.4) holds.

This paper is organized as follows. Sections 1–3 give results which were obtained in [KV] and [TV] (see also [BT]). Its major part is an exposition of the s-component KP hierarchy following [KV]. In Section 1, we describe the semi-infinite wedge representation of the group GL_∞ and the Lie algebras gl_∞ and a_∞. We define the KP hierarchy in the so-called fermionic picture. The loop algebra $\widehat{gl}_n$ is introduced in Section 2. We obtain it as a subalgebra of a_∞. Next we construct to every partition $n = n_1+n_2+\cdots+n_s$ of n a vertex operator realization of a_∞ and $\widehat{gl}_n$. Section 3 is devoted to the description of s-component KP hierarchy in terms of formal pseudo-differential operators. Section 4 describes reductions of this s-component KP hierarchy related to the above partitions. In Section 5 we introduce the string equation and deduce its consequences in terms of the pseudo-differential operators. Using the results of Section 5 we deduce in Section 6 the $W_{1+\infty}$ constraints. Section 7 is devoted to a geometric interpretation of the string equation on the Sato Grassmannian, which is similar to that of [KS].

Notice that, since the Toda lattice hierarchy of [UT] is related to the 2-component KP hierarchy, some results of this paper also hold for certain reductions of the Toda lattice hierarchy.

1. The semi-infinite wedge representation of the group GL_∞ and the KP hierarchy in the fermionic picture

1.1. Consider the infinite complex matrix group

$$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbb{Z}+1/2} \mid A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0\}$$

and its Lie algebra

$$gl_\infty = \{a = (a_{ij})_{i,j\in\mathbb{Z}+1/2} \mid \text{all but a finite number of } a_{ij} \text{ are } 0\}$$

with bracket $[a,b] = ab - ba$. This Lie algebra has a basis consisting of matrices E_{ij}, $i,j \in \mathbb{Z}+\frac{1}{2}$, where E_{ij} is the matrix with a 1 on the (i,j)th entry and zeros elsewhere. Now gl_∞ is a subalgebra of the bigger Lie algebra

$$\overline{gl_\infty} = \{a = (a_{ij})_{i,j\in\mathbb{Z}+1/2} \mid a_{ij} = 0 \text{ if } |i-j| \gg 0\}.$$

This Lie algebra $\overline{gl_\infty}$ has a universal central extension $a_\infty := \overline{gl_\infty} \bigoplus \mathbb{C}c$ with Lie bracket defined by

$$[a+\alpha c, b+\beta c] = ab - ba + \mu(a,b)c, \tag{1.1.1}$$

for $a,b \in \overline{gl_\infty}$ and $\alpha,\beta \in \mathbb{C}$; here μ is the following 2-cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}(\theta(i) - \theta(j)), \tag{1.1.2}$$

where $\theta : \mathbb{R} \to \mathbb{C}$ is defined by

$$\theta(i) := \begin{cases} 0 & \text{if } i > 0, \\ 1 & \text{if } i \le 0. \end{cases} \tag{1.1.3}$$

Let $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}+1/2}\mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis $\{v_j\}_{j\in\mathbb{Z}+1/2}$. Both the group GL_∞ and the Lie algebras gl_∞ and a_∞ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk}v_i.$$

We introduce, following [KP2], the corresponding semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$, this is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\cdots$, where $i_1 > i_2 > i_3 > \cdots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell \gg 0$. We can now define representations R of GL_∞ on F by

$$R(A)(v_{i_1}\wedge v_{i_2}\wedge v_{i_3}\wedge\cdots) = Av_{i_1}\wedge Av_{i_2}\wedge Av_{i_3}\wedge\cdots. \tag{1.1.4}$$

In order to describe representations of the Lie algebras we find it convenient to define wedging and contracting operators ψ_j^- and ψ_j^+ $(j\in\mathbb{Z}+\frac{1}{2})$ on F by

$$\psi_j^-(v_{i_1}\wedge v_{i_2}\wedge\cdots) = \begin{cases} 0 & \text{if } -j = i_s \text{ for some } s \\ (-1)^s v_{i_1}\wedge v_{i_2}\cdots\wedge v_{i_s}\wedge v_{-j}\wedge v_{i_{s+1}}\wedge\cdots & \text{if } i_s > -j > i_{s+1} \end{cases}$$

$$\psi_j^+(v_{i_1}\wedge v_{i_2}\wedge\cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s \\ (-1)^{s+1} v_{i_1}\wedge v_{i_2}\wedge\cdots\wedge v_{i_{s-1}}\wedge v_{i_{s+1}}\wedge\cdots & \text{if } j = i_s. \end{cases}$$

Notice that the definition of $\psi_j^\pm$ differs from the one in [KV]. The reason for this will become clear in Section 7, where we describe the connection with the Sato Grassmannian. These wedging and contracting operators satisfy the following relations $(i,j\in\mathbb{Z}+\frac{1}{2}, \lambda,\mu = +,-)$:

$$\psi_i^\lambda\psi_j^\mu + \psi_j^\mu\psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j}, \tag{1.1.5}$$

hence they generate a Clifford algebra, which we denote by $C\ell$.

Introduce the following elements of F $(m\in\mathbb{Z})$:

$$|m\rangle = v_{m-1/2}\wedge v_{m-3/2}\wedge v_{m-5/2}\wedge\cdots.$$

It is clear that F is an irreducible $C\ell$-module such that

$$\psi_j^\pm|0\rangle = 0 \quad \text{for } j>0. \tag{1.1.6}$$

We are now able to define representations r, $\hat{r}$ of gl_∞, a_∞ on F by

$$r(E_{ij}) = \psi_{-i}^-\psi_j^+, \qquad \hat{r}(E_{ij}) = :\psi_{-i}^-\psi_j^+:, \qquad \hat{r}(c) = I,$$

where : : stands for the *normal ordered product* defined in the usual way $(\lambda,\mu = +$ or $-)$:

$$:\psi_k^\lambda \psi_\ell^\mu: = \begin{cases} \psi_k^\lambda \psi_\ell^\mu & \text{if } \ell \geq k \\ -\psi_\ell^\mu \psi_k^\lambda & \text{if } \ell < k. \end{cases} \tag{1.1.7}$$

1.2. Define the *charge decomposition*

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)} \tag{1.2.1}$$

by letting

$$\text{charge}(|0\rangle) = 0 \quad \text{and} \quad \text{charge}(\psi_j^\pm) = \pm 1. \tag{1.2.2}$$

It is easy to see that each $F^{(m)}$ is irreducible with respect to $g\ell_\infty$, a_∞ (and GL_∞). Note that $|m\rangle$ is its highest weight vector, i.e., $\hat{r}(E_{ij}) = r(E_{ij}) - \delta_{ij}\theta(i)$ and

$$r(E_{ij})|m\rangle = 0 \quad \text{for } i < j,$$
$$r(E_{ii})|m\rangle = 0 \ (= |m\rangle) \quad \text{if } i > m \ (i < m).$$

Let $\mathcal{O} = R(GL_\infty)|0\rangle \subset F^{(0)}$ be the GL_∞-orbit of the vacuum vector $|0\rangle$, then one has

Proposition 1.1 ([KP2]). *A non-zero element τ of $F^{(0)}$ lies in $\mathcal{O}$ if and only if the following equation holds in $F \otimes F$:*

$$\sum_{k \in \mathbb{Z}+1/2} \psi_k^+ \tau \otimes \psi_{-k}^- \tau = 0. \tag{1.2.3}$$

Proof. For a proof see [KP2] or [KR]. □

Eq. (1.2.3) is called the *KP hierarchy in the fermionic picture.*

2. The loop algebra $\widehat{gl}_n$, partitions of n and vertex operator constructions

2.1. Let $\tilde{gl}_n = gl_n(\mathbb{C}[t,t^{-1}])$ be the loop algebra associated to $gl_n(\mathbb{C})$. This algebra has a natural representation on the vector space $(\mathbb{C}[t,t^{-1}])^n$. Let $\{w_i\}$ be the standard basis of $\mathbb{C}^n$. By identifying $(\mathbb{C}[t,t^{-1}])^n$ over $\mathbb{C}$ with $\mathbb{C}^\infty$ via $v_{nk+j-1/2} = t^{-k}w_j$ we obtain an embedding $\phi : \tilde{gl}_n \to \overline{gl_\infty}$:

$$\phi(t^k e_{ij}) = \sum_{\ell \in \mathbb{Z}} E_{n(\ell-k)+i-1/2, n\ell+j-1/2},$$

where e_{ij} is a basis of $gl_n(\mathbb{C})$.

A straightforward calculation shows that the restriction of the cocycle μ to $\phi(\tilde{gl}_n)$ induces the following 2-cocycle on $\tilde{gl}_n$:

$$\mu(x(t), y(t)) = \text{Res}_{t=0}\, dt \, \text{tr}\left(\frac{dx(t)}{dt} y(t)\right).$$

Here and further $\text{Res}_{t=0}\, dt \sum_j f_j t^j$ stands for f_{-1}. This gives a central extension $\widehat{gl}_n = \widetilde{gl}_n \bigoplus \mathbb{C}K$, where the bracket is defined by

$$[t^\ell x + \alpha K, t^m y + \beta K] = t^{\ell+m}(xy - yx) + \ell \delta_{\ell,-m} \text{tr}(xy) K.$$

In this way we have an embedding $\phi : \widehat{gl}_n \to a_\infty$, where $\phi(K) = c$.

Since F is a module for a_∞, it is clear that with this embedding we also have a representation of $\widehat{gl}_n$ on this semi-infinite wedge space. It is well-known that the level one representations of the affine Kac-Moody algebra $\widehat{gl}_n$ have a lot of inequivalent realizations. To be more precise, Kac and Peterson [KP1] and independently Lepowsky [L] showed that to every conjugacy class of the Weyl group of $gl_n(\mathbb{C})$ or rather $sl_n(\mathbb{C})$ there exists an inequivalent vertex operator realization of the same level one module. Hence to every partition of n, there exists such a construction.

We will now sketch how one can construct these vertex realizations of $\widehat{gl}_n$, following [TV]. From now on let $n = n_1 + n_2 + \cdots + n_s$ be a partition of n into s parts, and denote by $N_a = n_1 + n_2 + \cdots + n_{a-1}$. We begin by relabeling the basis vectors v_j and with them the corresponding fermionic (wedging and contracting) operators: $(1 \le a \le s,\ 1 \le p \le n_a,\ j \in \mathbb{Z})$

$$\begin{aligned} &v^{(a)}_{n_a j - p + 1/2} = v_{nj - N_a - p + 1/2}, \\ &\psi^{\pm(a)}_{n_a j \mp p \pm 1/2} = \psi^{\pm}_{nj \mp N_a \mp p \pm 1/2}. \end{aligned} \tag{2.1.1}$$

Notice that with this relabeling we have: $\psi^{\pm(a)}_k |0\rangle = 0$ for $k > 0$. We also rewrite the E_{ij}'s:

$$E^{(ab)}_{n_a j - p + 1/2, n_b k - q + 1/2} = E_{nj - N_a - p + 1/2, nk - N_b - q + 1/2}.$$

The corresponding Lie bracket on a_∞ is given by

$$[E^{(ab)}_{jk}, E^{(cd)}_{\ell m}] = \delta_{bc}\delta_{kl} E^{(ad)}_{jm} - \delta_{ad}\delta_{jm} E^{(db)}_{\ell k} + \delta_{ad}\delta_{bc}\delta_{jm}\delta_{k\ell}(\theta(j) - \theta(k))c,$$

and $\hat{r}(E^{(ab)}_{jk}) = :\psi^{-(a)}_{-j} \psi^{+b}_k:$.

Introduce the fermionic fields $(z \in \mathbb{C}^\times)$:

$$\psi^{\pm(a)}(z) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}+1/2} \psi^{\pm(a)}_k z^{-k-1/2}. \tag{2.1.2}$$

Let N be the least common multiple of $n_1, n_2, \ldots, n_s$. It was shown in [TV] that the modes of the fields

$$:\psi^{+(a)}(\omega_a^p z^{N/n_a}) \psi^{-(b)}(\omega_b^q z^{N/n_b}):, \tag{2.1.3}$$

for $1 \le a, b \le s$, $1 \le p \le n_a$, $1 \le q \le n_b$, where $\omega_a = e^{2\pi i/n_a}$, together with the identity, generate a representation of $\widehat{gl}_n$ with $K = 1$.

Next we introduce special bosonic fields $(1 \le a \le s)$:

$$\alpha^{(a)}(z) \equiv \sum_{k \in \mathbb{Z}} \alpha^{(a)}_k z^{-k-1} \stackrel{\text{def}}{=} :\psi^{+(a)}(z) \psi^{-(a)}(z):. \tag{2.1.4}$$

The operators $\alpha_k^{(a)}$ satisfy the canonical commutation relation of the associative oscillator algebra, which we denote by $\mathfrak{a}$:

$$[\alpha_k^{(i)}, \alpha_\ell^{(j)}] = k\delta_{ij}\delta_{k,-\ell}, \tag{2.1.5}$$

and one has

$$\alpha_k^{(i)}|m\rangle = 0 \quad \text{for } k > 0. \tag{2.1.6}$$

It is easy to see that restricted to $\widehat{gl}_n$, $F^{(0)}$ is its basic highest weight representation (see [K, Ch. 12]).

In order to express the fermionic fields $\psi^{\pm(i)}(z)$ in terms of the bosonic fields $\alpha^{(i)}(z)$, we need some additional operators Q_i, $i = 1, \dots, s$, on F. These operators are uniquely defined by the following conditions:

$$Q_i|0\rangle = \psi_{-1/2}^{+(i)}|0\rangle, \quad Q_i\psi_k^{\pm(j)} = (-1)^{\delta_{ij}+1}\psi_{k\mp\delta_{ij}}^{\pm(j)}Q_i. \tag{2.1.7}$$

They satisfy the following commutation relations:

$$Q_iQ_j = -Q_jQ_i \quad \text{if } i \neq j, \qquad [\alpha_k^{(i)}, Q_j] = \delta_{ij}\delta_{k0}Q_j. \tag{2.1.8}$$

Theorem 2.1 ([DJKM1, JM]).

$$\psi^{\pm(i)}(z) = Q_i^{\pm 1} z^{\pm\alpha_0^{(i)}} \exp\left(\mp\sum_{k<0}\frac{1}{k}\alpha_k^{(i)}z^{-k}\right)\exp\left(\mp\sum_{k>0}\frac{1}{k}\alpha_k^{(i)}z^{-k}\right). \tag{2.1.9}$$

Proof. See [TV].

The operators on the right-hand side of (2.1.9) are called vertex operators. They made their first appearance in string theory (cf. [FK]).

If one substitutes (2.1.9) into (2.1.3), one obtains the vertex operator realization of $\widehat{gl}_n$, which is related to the partition $n = n_1 + n_2 + \cdots + n_s$ (see [TV] for more details).

2.2. The realization of $\widehat{gl}_n$, described in the previous section, has a natural Virasoro algebra. In [TV], it was shown that the following two sets of operators have the same action on F:

$$L_k = \sum_{i=1}^{s}\left\{\sum_{j\in\mathbb{Z}}\frac{1}{2n_i}:\alpha_{-j}^{(i)}\alpha_{j+n_ik}^{(i)}: + \delta_{k0}\frac{n_i^2-1}{24n_i}\right\}, \tag{2.2.1}$$

$$H_k = \sum_{i=1}^{s}\left\{\sum_{j\in\mathbb{Z}+1/2}\left(\frac{j}{n_i}+\frac{k}{2}\right):\psi_{-j}^{+(i)}\psi_{j+n_ik}^{-(i)}: + \delta_{k0}\frac{n_i^2-1}{24n_i}\right\}. \tag{2.2.2}$$

So $L_k = H_k$,

$$[L_k, \psi_j^{\pm(i)}] = -\left(\frac{j}{n_i}+\frac{k}{2}\right)\psi_{j+n_ik}^{\pm(i)} \tag{2.2.3}$$

and

$$[L_k, L_\ell] = (k-\ell)L_{k+\ell} + \delta_{k,-\ell}\frac{k^3-k}{12}n.$$

2.3. We will now use the results of Section 2.1 to describe the s-component boson–fermion correspondence. Let $\mathbb{C}[x]$ be the space of polynomials in indeterminates $x = \{x_k^{(i)}\}$, $k = 1, 2, \ldots,$ $i = 1, 2, \ldots, s$. Let L be a lattice with a basis $\delta_1, \ldots, \delta_s$ over $\mathbb{Z}$ and the symmetric bilinear form $(\delta_i|\delta_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Let

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \le j. \end{cases} \tag{2.3.1}$$

Define a bimultiplicative function $\varepsilon : L \times L \to \{\pm 1\}$ by letting

$$\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}. \tag{2.3.2}$$

Let $\delta = \delta_1 + \cdots + \delta_s$, $Q = \{\gamma \in L \mid (\delta|\gamma) = 0\}$, $\Delta = \{\alpha_{ij} := \delta_i - \delta_j \mid i, j = 1, \ldots, s,\ i \ne j\}$. Of course Q is the root lattice of $sl_s(\mathbb{C})$, the set Δ being the root system.

Consider the vector space $\mathbb{C}[L]$ with basis e^γ, $\gamma \in L$, and the following twisted group algebra product:

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}. \tag{2.3.3}$$

Let $B = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[L]$ be the tensor product of algebras. Then the s-component boson–fermion correspondence is the vector space isomorphism

$$\sigma : F \xrightarrow{\sim} B, \tag{2.3.4}$$

given by

$$\sigma(\alpha_{-m_1}^{(i_1)} \cdots \alpha_{-m_r}^{(i_r)} Q_1^{k_1} \cdots Q_s^{k_s}|0\rangle) = m_1 \cdots m_s x_{m_1}^{(i_1)} \cdots x_{m_r}^{(i_r)} \otimes e^{k_1\delta_1 + \cdots + k_s\delta_s}. \tag{2.3.5}$$

The transported charge then will be as follows:

$$\text{charge}(p(x) \otimes e^\gamma) = (\delta|\gamma). \tag{2.3.6}$$

We denote the transported charge decomposition by

$$B = \bigoplus_{m\in\mathbb{Z}} B^{(m)}.$$

The transported action of the operators $\alpha_m^{(i)}$ and Q_j looks as follows:

$$\begin{aligned}
\sigma\alpha_{-m}^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) &= m x_m^{(j)} p(x) \otimes e^\gamma, \quad \text{if } m > 0, \\
\sigma\alpha_m^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) &= \partial p(x)/\partial x_m^{(j)} \otimes e^\gamma, \quad \text{if } m > 0, \\
\sigma\alpha_0^{(j)}\sigma^{-1}(p(x) \otimes e^\gamma) &= (\delta_j|\gamma)p(x) \otimes e^\gamma, \\
\sigma Q_j\sigma^{-1}(p(x) \otimes e^\gamma) &= \varepsilon(\delta_j, \gamma)p(x) \otimes e^{\gamma+\delta_j}.
\end{aligned} \tag{2.3.7}$$

For notational convenience, we introduce $\delta_j = \sigma\alpha_0^{(j)}\sigma^{-1}$. Notice that $e^{\delta_j} = \sigma Q_j\sigma^{-1}$.

2.4. Using the isomorphism σ we can reformulate the KP hierarchy (1.2.3) in the bosonic picture. We start by observing that (1.2.3) can be rewritten as follows:

$$\operatorname{Res}_{z=0} dz \left(\sum_{j=1}^{s} \psi^{+(j)}(z)\tau \otimes \psi^{-(j)}(z)\tau \right) = 0, \quad \tau \in F^{(0)}. \tag{2.4.1}$$

Notice that for $\tau \in F^{(0)}$, $\sigma(\tau) = \sum_{\gamma \in Q} \tau_\gamma(x) e^\gamma$. Here and further we write $\tau_\gamma(x)e^\gamma$ for $\tau_\gamma(x) \otimes e^\gamma$. Using Theorem 2.1, Eq. (2.4.1) turns under $\sigma \otimes \sigma : F \otimes F \xrightarrow{\sim} \mathbb{C}[x', x''] \otimes (\mathbb{C}[L'] \otimes \mathbb{C}[L''])$ into the following set of equations: for all $\alpha, \beta \in L$ such that $(\alpha|\delta) = -(\beta|\delta) = 1$ we have

$$\begin{aligned} &\operatorname{Res}_{z=0} \left(dz \sum_{j=1}^{s} \varepsilon(\delta_j, \alpha - \beta) z^{(\delta_j|\alpha-\beta-2\delta_j)} \right. \\ &\times \exp\left(\sum_{k=1}^{\infty} (x_k^{(j)\prime} - x_k^{(j)\prime\prime}) z^k \right) \exp\left(-\sum_{k=1}^{\infty} \left(\frac{\partial}{\partial x_k^{(j)\prime}} - \frac{\partial}{\partial x_k^{(j)\prime\prime}} \right) \frac{z^{-k}}{k} \right) \\ &\left. \times \tau_{\alpha-\delta_j}(x')(e^\alpha)' \tau_{\beta+\delta_j}(x'')(e^\beta)'' \right) = 0. \end{aligned} \tag{2.4.2}$$

3. The algebra of formal pseudo-differential operators and the s-component KP hierarchy as a dynamical system

3.0. The KP hierarchy and its s-component generalizations admit several formulations. The one we will give here was introduced by Sato [S]; it is given in the language of formal pseudo-differential operators. We will show that this formulation follows from the τ-function formulation given by Eq. (2.4.2).

3.1. We shall work over the algebra $\mathcal{A}$ of formal power series over $\mathbb{C}$ in indeterminates $x = (x_k^{(j)})$, where $k = 1, 2, \ldots$ and $j = 1, \ldots, s$. The indeterminates $x_1^{(1)}, \ldots, x_1^{(s)}$ will be viewed as variables and $x_k^{(j)}$ with $k \geq 2$ as parameters. Let

$$\partial = \frac{\partial}{\partial x_1^{(1)}} + \cdots + \frac{\partial}{\partial x_1^{(s)}}.$$

A formal $s \times s$ matrix pseudo-differential operator is an expression of the form

$$P(x, \partial) = \sum_{j \leq N} P_j(x) \partial^j, \tag{3.1.1}$$

where P_j are $s \times s$ matrices over $\mathcal{A}$. Let Ψ denote the vector space over $\mathbb{C}$ of all expressions (3.1.1). We have a linear isomorphism $S : \Psi \to \operatorname{Mat}_s(\mathcal{A}((z)))$ given by $S(P(x, \partial)) = P(x, z)$. The matrix series $P(x, z)$ in indeterminates x and z is called the *symbol* of $P(x, \partial)$.

Now we may define a product $\circ$ on Ψ making it an associative algebra:

$$S(P \circ Q) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n S(P)}{\partial z^n} \partial^n S(Q). \tag{3.1.2}$$

From now on, we shall drop the multiplication sign $\circ$ when no ambiguity may arise. One defines the differential part of $P(x,\partial)$ by $P_+(x,\partial) = \sum_{j=0}^{N} P_j(x)\partial^j$, and let $P_- = P - P_+$. We have the corresponding vector space decomposition:

$$\Psi = \Psi_- \oplus \Psi_+. \tag{3.1.3}$$

One defines a linear map $* : \Psi \to \Psi$ by the following formula:

$$\left(\sum_j P_j \partial^j\right)^* = \sum_j (-\partial)^j \circ {}^t P_j. \tag{3.1.4}$$

Here and further ${}^tP$ stands for the transpose of the matrix P. Note that $*$ is an anti-involution of the algebra Ψ.

3.2. Introduce the following notation:

$$z \cdot x^{(j)} = \sum_{k=1}^{\infty} x_k^{(j)} z^k, \quad e^{z \cdot x} = \mathrm{diag}(e^{z \cdot x^{(1)}}, \ldots, e^{z \cdot x^{(s)}}).$$

The algebra Ψ acts on the space U_+ (U_-) of formal oscillating matrix functions of the form

$$\sum_{j \le N} P_j z^j e^{z \cdot x} \left(\sum_{j \le N} P_j z^j e^{-z \cdot x}\right), \quad \text{where } P_j \in \mathrm{Mat}_s(\mathcal{A}),$$

in the obvious way:

$$P(x)\partial^j e^{\pm z \cdot x} = P(x)(\pm z)^j e^{\pm z \cdot x}.$$

One has the following fundamental lemma (see [KV]).

Lemma 3.1. *If $P, Q \in \Psi$ are such that*

$$\mathrm{Res}_{z=0}(P(x,\partial)e^{z \cdot x})\, {}^t(Q(x',\partial')e^{-z \cdot x'})\, dz = 0, \tag{3.2.1}$$

then $(P \circ Q^*)_- = 0$.

3.3. We proceed now by rewriting the formulation (2.4.2) of the s-component KP hierarchy in terms of formal pseudo-differential operators.

For each $\alpha \in \operatorname{supp} \tau := \{\alpha \in Q \mid \tau = \sum_{\alpha \in Q} \tau_\alpha e^\alpha, \tau_\alpha \neq 0\}$ we define the (matrix valued) functions

$$V^{\pm}(\alpha, x, z) = (V_{ij}^{\pm}(\alpha, x, z))_{i,j=1}^{s} \tag{3.3.1}$$

as follows:

$$V_{ij}^{\pm}(\alpha,x,z) \stackrel{\text{def}}{=} \varepsilon(\delta_j,\alpha+\delta_i)z^{(\delta_j|\pm\alpha+\delta_i-\delta_j)}(\tau_\alpha(x))^{-1} \times \exp\left(\pm\sum_{k=1}^{\infty} x_k^{(j)}z^k\right)\exp\left(\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k}\right)\tau_{\alpha\pm(\delta_i-\delta_j)}(x). \tag{3.3.2}$$

It is easy to see that Eq. (2.4.2) is equivalent to the following bilinear identity:

$$\mathrm{Res}_{z=0}\, V^+(\alpha,x,z)\ {}^tV^-(\beta,x',z)\,dz = 0 \quad \text{for all } \alpha,\beta\in Q. \tag{3.3.3}$$

Define $s\times s$ matrices $W^{\pm(m)}(\alpha,x)$ by the following generating series (cf. (3.3.2)):

$$\sum_{m=0}^{\infty} W_{ij}^{\pm(m)}(\alpha,x)(\pm z)^{-m} = \varepsilon_{ji}z^{\delta_{ij}-1}(\tau_\alpha(x))^{-1}\left(\exp\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k}\right)\tau_{\alpha\pm\alpha_{ij}}(x)). \tag{3.3.4}$$

We see from (3.3.2) that $V^\pm(\alpha,x,z)$ can be written in the following form:

$$V^\pm(\alpha,x,z) = \left(\sum_{m=0}^{\infty} W^{\pm(m)}(\alpha,x)R^\pm(\alpha,\pm z)(\pm z)^{-m}\right)e^{\pm z\cdot x}, \tag{3.3.5}$$

where

$$R^\pm(\alpha,z) = \sum_{i=1}^{s}\varepsilon(\delta_i,\alpha)E_{ii}(\pm z)^{\pm(\delta_i|\alpha)}. \tag{3.3.6}$$

Here and further E_{ij} stands for the $s\times s$ matrix whose (i,j) entry is 1 and all other entries are zero. Now it is clear that $V^\pm(\alpha,x,z)$ can be written in terms of formal pseudo-differential operators

$$P^\pm(\alpha)\equiv P^\pm(\alpha,x,\partial) = I_s + \sum_{m=1}^{\infty} W^{\pm(m)}(\alpha,x)\partial^{-m},$$
$$R^\pm(\alpha) = R^\pm(\alpha,\partial) \tag{3.3.7}$$

as follows:

$$V^\pm(\alpha,x,z) = P^\pm(\alpha)R^\pm(\alpha)e^{\pm z\cdot x}. \tag{3.3.8}$$

Since obviously $R^-(\alpha,\partial)^{-1} = R^+(\alpha,\partial)^*$, using Lemma 3.1 we deduce from the bilinear identity (3.3.3):

$$P^-(\alpha) = (P^+(\alpha)^*)^{-1}, \tag{3.3.9}$$

$$(P^+(\alpha)R^+(\alpha-\beta)P^+(\beta)^{-1})_- = 0 \quad \text{for all } \alpha,\beta\in \mathrm{supp}\,\tau. \tag{3.3.10}$$

Victor Kac and the author showed in [KV] that given $\beta \in \operatorname{supp} \tau$, all the pseudo-differential operators $P^+(\alpha)$, $\alpha \in \operatorname{supp} \tau$, are completely determined by $P^+(\beta)$ from Eqs. (3.3.10). They also showed that $P = P^+(\alpha)$ satisfies the Sato equation:

$$\frac{\partial P}{\partial x_k^{(j)}} = -(PE_{jj} \circ \partial^k \circ P^{-1})_- \circ P. \tag{3.3.11}$$

To be more precise, one has the following

Proposition 3.2. *Consider the formal oscillating functions $V^+(\alpha, x, z)$, $V^-(\alpha, x, z)$, $\alpha \in Q$, of the form (3.3.8), where $R^\pm(\alpha, z)$ are given by (3.3.6) and $P^\pm(\alpha, x, \partial) \in I_s + \Psi_-$. Then the bilinear identity (3.3.3) for all $\alpha, \beta \in$ supp τ is equivalent to the Sato equation (3.3.11) for each $P = P^+(\alpha)$ and the matching conditions (3.3.9), (3.3.10) for all $\alpha, \beta \in$ supp τ.*

3.4. Fix $\alpha \in Q$, introduce the following formal pseudo-differential operators $L(\alpha)$, $C^{(j)}(\alpha)$, and differential operators $B_m^{(j)}(\alpha)$:

$$\begin{aligned} L \equiv L(\alpha) &= P^+(\alpha) \circ \partial \circ P^+(\alpha)^{-1}, \\ C^{(j)} \equiv C^{(j)}(\alpha) &= P^+(\alpha) E_{jj} P^+(\alpha)^{-1}, \\ B_m^{(j)} \equiv B_m^{(j)}(\alpha) &= (P^+(\alpha) E_{jj} \circ \partial^m \circ P^+(\alpha)^{-1})_+. \end{aligned} \tag{3.4.1}$$

Then

$$\begin{aligned} L &= I_s \partial + \sum_{j=1}^{\infty} U^{(j)}(x) \partial^{-j}, \\ C^{(i)} &= E_{ii} + \sum_{j=1}^{\infty} C^{(i,j)}(x) \partial^{-j}, \quad i = 1, 2, \cdots, s, \end{aligned} \tag{3.4.2}$$

subject to the conditions

$$\sum_{i=1}^{s} C^{(i)} = I_s, \quad C^{(i)} L = L C^{(i)}, \quad C^{(i)} C^{(j)} = \delta_{ij} C^{(i)}. \tag{3.4.3}$$

They satisfy the following set of equations for some $P \in I_s + \Psi_-$:

$$\begin{aligned} LP &= P\partial, \\ C^{(i)} P &= P E_{ii}, \\ \partial P / \partial x_k^{(i)} &= -(L^{(i)k})_- P, \quad \text{where } L^{(i)} = C^{(i)} L. \end{aligned} \tag{3.4.4}$$

Proposition 3.3. *The system of equations (3.4.4) has a solution $P \in I_s + \Psi_-$ if and only if we can find a formal oscillating function of the form*

$$W(x, z) = \left(I_s + \sum_{j=1}^{\infty} W^{(j)}(x) z^{-j} \right) e^{z \cdot x} \tag{3.4.5}$$

that satisfies the linear equations

$$LW = zW, \quad C^{(i)}W = WE_{ii}, \quad \frac{\partial W}{\partial x_k^{(i)}} = B_k^{(i)}W. \tag{3.4.6}$$

And finally, one has the following

Proposition 3.4. *If for every $\alpha \in Q$ the formal pseudo-differential operators $L \equiv L(\alpha)$ and $C^{(j)} \equiv C^{(j)}(\alpha)$ of the form (3.4.2) satisfy conditions (3.4.3) and if Eqs. (3.4.4) have a solution $P \equiv P(\alpha) \in I_s + \Psi_-$, then the differential operators $B_k^{(j)} \equiv B_k^{(j)}(\alpha)$ satisfy one of the following equivalent conditions:*

$$\frac{\partial L}{\partial x_k^{(j)}} = [B_k^{(j)}, L], \quad \frac{\partial C^{(i)}}{\partial x_k^{(j)}} = [B_k^{(j)}, C^{(i)}], \tag{3.4.7}$$

$$\frac{\partial L^{(i)}}{\partial x_k^{(j)}} = [B_k^{(j)}, L^{(i)}], \tag{3.4.8}$$

$$\frac{\partial B_\ell^{(i)}}{\partial x_k^{(j)}} - \frac{\partial B_k^{(j)}}{\partial x_\ell^{(i)}} = [B_k^{(j)}, B_\ell^{(i)}]. \tag{3.4.9}$$

Here $L^{(j)} \equiv L^{(j)}(\alpha) = C^{(j)}(\alpha) \circ L(\alpha)$.

Eqs. (3.4.7) and (3.4.8) are called *Lax type* equations. Eqs. (3.4.9) are called the *Zakharov–Shabat type* equations. The latter are the compatibility conditions for the linear problem (3.4.6).

4. $[n_1, n_2, \ldots, n_s]$-reductions of the s-component KP hierarchy

4.1. Using (2.1.9), (2.1.3), (2.3.5) and (2.3.7), we obtain the vertex operator realization of $\widehat{gl}_n$ in the vector space $B^{(m)}$ that is related to the partition $n = n_1 + n_2 + \cdots + n_s$. Now, restricted to $\widehat{sl}_n$, the representation in $F^{(m)}$ is not irreducible anymore, since $\widehat{sl}_n$ commutes with the operators

$$\beta_{kn_s}^{(s)} = \sqrt{\frac{n_s}{N}} \sum_{i=1}^{s} \alpha_{kn_i}^{(i)}, \quad k \in \mathbb{Z}. \tag{4.1.1}$$

In order to describe the irreducible part of the representation of $\widehat{sl}_n$ in $B^{(0)}$ containing the vacuum vector 1, we choose the complementary generators of the oscillator algebra $\mathfrak{a}$ contained in $\widehat{sl}_n$ $(k \in \mathbb{Z})$:

$$\beta_k^{(j)} = \begin{cases} \alpha_k^{(j)} & \text{if } k \notin n_j\mathbb{Z}, \\ \dfrac{N_{j+1}\alpha_{\ell n_{j+1}}^{(j+1)} - n_{j+1}(\alpha_{\ell n_1}^{(1)} + \alpha_{\ell n_2}^{(2)} + \cdots + \alpha_{\ell n_j}^{(j)})}{\sqrt{N_{j+1}(N_{j+1} - n_{j+1})}} & \text{if } k = \ell n_j \text{ and } 1 \le j < s, \end{cases} \tag{4.1.2}$$

so that the operators (4.1.1) and (4.1.2) also satisfy relations (2.1.5). Hence, introducing the new indeterminates

$$y_k^{(j)} = \begin{cases} x_k^{(j)} & \text{if } k \notin n_j\mathbb{N}, \\ \dfrac{N_{j+1}x_{\ell n_{j+1}}^{(j+1)} - (n_1 x_{\ell n_1}^{(1)} + n_2 x_{\ell n_2}^{(2)} + \cdots + n_j x_{\ell n_j}^{(j)})}{\sqrt{N_{j+1}(N_{j+1} - n_{j+1})}} & \text{if } k = \ell n_j \text{ and } 1 \le j < s, \\ \dfrac{n_1 x_{\ell n_1}^{(1)} + n_2 x_{\ell n_2}^{(2)} + \cdots + n_s x_{\ell n_s}^{(s)}}{\sqrt{N n_s}} & \text{if } k = \ell n_s \text{ and } j = s, \end{cases} \tag{4.1.3}$$

we have $\mathbb{C}[x] = \mathbb{C}[y]$ and

$$\sigma(\beta_k^{(j)}) = \partial/\partial y_k^{(j)} \quad \text{and} \quad \sigma(\beta_{-k}^{(j)}) = k y_k^{(j)} \quad \text{if } k > 0. \tag{4.1.4}$$

Now it is clear that the subspace of $B^{(0)}$ irreducible with respect to $\widehat{sl}_n$ and containing the vacuum 1 is the vector space

$$B^{(0)}_{[n_1,n_2,\ldots,n_s]} = \mathbb{C}[y_k^{(j)} \mid 1 \le j < s,\ k \in \mathbb{N},\ \text{or } j = s,\ k \in \mathbb{N}\backslash n_s\mathbb{Z}] \otimes \mathbb{C}[Q]. \tag{4.1.5}$$

The vertex operator realization of $\widehat{sl}_n$ in the vector space $B^{(0)}_{[n_1,n_2,\ldots,n_s]}$ is then obtained by expressing the fields (2.1.3) in terms of vertex operators (2.1.9), which are expressed via (4.1.2) in the operators (4.1.4), the operators $e^{\delta_i - \delta_j}$ and $\delta_i - \delta_j$ $(1 \le i < j \le s)$ (see [TV] for details).

The s-component KP hierarchy of Eqs. (2.4.2) on $\tau \in B^{(0)} = \mathbb{C}[y] \otimes \mathbb{C}[Q]$ when restricted to $\tau \in B^{(0)}_{[n_1,n_2,\ldots,n_s]}$ is called the $[n_1, n_2, \ldots, n_s]$*th reduced* KP *hierarchy*. It is obtained from the s-component KP hierarchy by making the change of variables (4.1.3) and putting zero all terms containing partial derivatives by $y_{n_s}^{(s)}, y_{2n_s}^{(s)}, y_{3n_s}^{(s)}, \ldots$.

The totality of solutions of the $[n_1, n_2, \ldots, n_s]$th reduced KP hierarchy is given by the following

Proposition 4.1. *Let $\mathcal{O}_{[n_1,n_2,\ldots,n_s]}$ be the orbit of* 1 *under the (projective) representation of the loop group $SL_n(\mathbb{C}[t, t^{-1}])$ corresponding to the representation of $\widehat{sl}_n$ in $B^{(0)}_{[n_1,n_2,\ldots,n_s]}$. Then*

$$\mathcal{O}_{[n_1,n_2,\ldots,n_s]} = \sigma(\mathcal{O}) \cap B^{(0)}_{[n_1,n_2,\ldots,n_s]}.$$

In other words, the τ-functions of the $[n_1, n_2, \ldots, n_s]$th reduced KP hierarchy are precisely the τ-functions of the KP hierarchy in the variables $y_k^{(j)}$, which are independent of the variables $y_{\ell n_s}^{(s)}$, $\ell \in \mathbb{N}$.

Proof. The same as the proof of a similar statement in [KP2]. □

4.2. It is clear from the definitions and results of Section 4.1 that the condition on the s-component KP hierarchy to be $[n_1, n_2, \ldots, n_s]$th reduced is equivalent to

$$\sum_{j=1}^{s} \frac{\partial \tau}{\partial x_{kn_j}^{(j)}} = 0, \quad \text{for all } k \in \mathbb{N} \tag{4.2.1}$$

Using the Sato equation (3.3.11), this implies the following two equivalent conditions:

$$\sum_{j=1}^{s} \frac{\partial W(\alpha)}{\partial x_{kn_j}^{(j)}} = W(\alpha) \sum_{j=1}^{s} z^{kn_j} E_{jj}, \tag{4.2.2}$$

$$\left(\sum_{j=1}^{s} L(\alpha)^{kn_j} C^{(j)} \right)_- = 0. \tag{4.2.3}$$

5. The string equation

5.1. From now on we assume that τ is any solution of the KP hierarchy. In particular, we no longer assume that τ_α is a polynomial. For instance, the soliton and dromion solutions of [KV, §5] are allowed. Of course this means that the corresponding wave functions $V^{\pm}(\alpha, z)$ will be of a more general nature than before.

Recall from Section 3 the wave function $V(\alpha, z) \equiv V^+(\alpha, z) = P(\alpha)R(\alpha)e^{z\cdot x} = P^+(\alpha)R^+(\alpha)e^{z\cdot x}$. It is natural to compute

$$\begin{aligned} \frac{\partial V(\alpha, z)}{\partial z} &= \frac{\partial}{\partial z} P(\alpha)R(\alpha)e^{z\cdot x} \\ &= P(\alpha)R(\alpha)\frac{\partial}{\partial z} e^{z\cdot x} \\ &= P(\alpha)R(\alpha) \sum_{a=1}^{s} \sum_{k=1}^{\infty} k x_k^{(a)} \partial^{k-1} E_{aa} R(\alpha)^{-1} P(\alpha)^{-1} V(\alpha, z). \end{aligned}$$

Define

$$M(\alpha) := P(\alpha)R(\alpha) \sum_{a=1}^{s} \sum_{k=1}^{\infty} k x_k^{(a)} \partial^{k-1} E_{aa} R(\alpha)^{-1} P(\alpha)^{-1}; \tag{5.1.1}$$

then one easily checks that $[L(\alpha), M(\alpha)] = 1$ and

$$\left[\sum_{a=1}^{s} L(\alpha)^{n_a} C^{(a)}(\alpha), M(\alpha) \sum_{a=1}^{s} \frac{1}{n_a} L(\alpha)^{1-n_a} C^{(a)}(\alpha) \right] = 1. \tag{5.1.2}$$

Next, we calculate the (i,j)th coefficient of

$$\left(M(\alpha) \sum_{a=1}^{s} \frac{1}{n_a} L(\alpha)^{1-n_a} C^{(a)}(\alpha) \right)_- P(\alpha)R(\alpha).$$

Let $P = S(P(\alpha))$ and $R = S(R(\alpha))$; then

$$\left(S\left(\left(M(\alpha)\sum_{a=1}^{s}\frac{1}{n_a}L(\alpha)^{1-n_a}C^{(a)}(\alpha)\right)_- P(\alpha)R(\alpha)\right)\right)_{ij}$$
$$=\left(S\left(\left(P(\alpha)R(\alpha)\sum_{a=1}^{s}\sum_{k\in\mathbb{N}}\frac{1}{n_a}kx_k^{(a)}\partial^{k-n_a}E_{aa}R(\alpha)^{-1}P(\alpha)^{-1}\right)_- P(\alpha)R(\alpha)\right)\right)_{ij}$$
$$=\left(\frac{\partial PR}{\partial z}\sum_{a=1}^{s}\frac{1}{n_a}E_{aa}z^{1-n_a}+\sum_{a=1}^{s}\left\{\sum_{k=1}^{n_a}\frac{k}{n_a}x_k^{(a)}PRE_{aa}z^{k-n_a}-x_{n_a}^{(a)}E_{aa}PR\right.\right.$$
$$\left.\left.-\sum_{k=1}^{\infty}\frac{k+n_a}{n_a}x_{k+n_a}^{(a)}\frac{\partial PR}{\partial x_k^{(a)}}\right\}\right)_{ij}.$$

Define

$$\tilde\tau_{\alpha+\delta_i-\delta_j}=\exp\left(-\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}\frac{z^{-k}}{k}\right)\tau_{\alpha+\delta_i-\delta_j}(x)$$
$$=\tau_{\alpha+\delta_i-\delta_j}(\ldots,x_k^{(b)}-\delta_{jb}/kz^k,\ldots);$$

then

$$(PR)_{ij}=\epsilon(\delta_j|\alpha+\delta_i)z^{\delta_{ij}-1+(\delta_j|\alpha)}\frac{\tilde\tau_{\alpha+\delta_i-\delta_j}}{\tau_\alpha}$$

and hence

$$\left(S\left(\left(M(\alpha)\sum_{a=1}^{s}\frac{1}{n_a}L(\alpha)^{1-n_a}C^{(a)}(\alpha)\right)_- P(\alpha)R(\alpha)\right)\right)_{ij}$$
$$=\frac{\epsilon(\delta_j|\alpha+\delta_i)}{n_j}\left\{\frac{1}{\tau_\alpha}\left(\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}z^{-k-n_j}+(\delta_{ij}-1+(\delta_j|\alpha))z^{-n_j}\right)\tilde\tau_{\alpha+\delta_i-\delta_j}\right.$$
$$+\sum_{k=1}^{n_j}kx_k^{(j)}\frac{\tilde\tau_{\alpha+\delta_i-\delta_j}}{\tau_\alpha}z^{k-n_j}-n_jx_{n_i}^{(i)}\frac{\tilde\tau_{\alpha+\delta_i-\delta_j}}{\tau_\alpha}$$
$$\left.-\sum_{a=1}^{s}\frac{n_j}{n_a}\sum_{k=1}^{\infty}(k+n_a)x_{k+n_a}^{(a)}\frac{\partial}{\partial x_k^{(a)}}\left(\frac{\tilde\tau_{\alpha+\delta_i-\delta_j}}{\tau_\alpha}\right)\right\}z^{\delta_{ij}-1+(\delta_j|\alpha)}. \tag{5.1.3}$$

5.2. We introduce the natural generalization of the string Eq. (0.3). Let L_{-1} be given by (2.2.1); the string equation is the following constraint on $\tau\in F^{(0)}$:

$$L_{-1}\tau=0. \tag{5.2.1}$$

Using (2.3.7) we rewrite L_{-1} in terms of operators on $B^{(0)}$:

$$L_{-1}=\sum_{a=1}^{s}\left\{\delta_a x_{n_a}^{(a)}+\frac{1}{2n_a}\sum_{p=1}^{n_a-1}p(n_a-p)x_p^{(a)}x_{n_a-p}^{(a)}\right.$$
$$\left.+\frac{1}{n_a}\sum_{k=1}^{\infty}(k+n_a)x_{k+n_a}^{(a)}\frac{\partial}{\partial x_k^{(a)}}\right\}.$$

Since $\tau=\sum_{\alpha\in Q}\tau_\alpha e^\alpha$ and $L_{-1}\tau=0$, we find that for all $\alpha\in Q$:

$$\sum_{a=1}^{s}\left\{(\delta_a|\alpha)x_{n_a}^{(a)}+\frac{1}{2n_a}\sum_{p=1}^{n_a-1}p(n_a-p)x_p^{(a)}x_{n_a-p}^{(a)}\right.$$
$$\left.+\frac{1}{n_a}\sum_{k=1}^{\infty}(k+n_a)x_{k+n_a}^{(a)}\frac{\partial}{\partial x_k^{(a)}}\right\}\tau_\alpha=0. \tag{5.2.2}$$

Clearly, also $L_{-1}\tilde{\tau}_{\alpha+\delta_i-\delta_j}=0$; this gives (see e.g. [D]):

$$\sum_{a=1}^{s}\left\{(\delta_a|\alpha+\delta_i-\delta_j)\left(x_{n_a}^{(a)}-\frac{\delta_{aj}}{n_jz^{n_j}}\right)\right.$$
$$+\frac{1}{2n_a}\sum_{p=1}^{n_a-1}p(n_a-p)\left(x_p^{(a)}-\frac{\delta_{aj}}{pz^p}\right)\left(x_{n_a-p}^{(a)}-\frac{\delta_{aj}}{(n_j-p)z^{n_j-p}}\right)$$
$$\left.+\frac{1}{n_a}\sum_{k=1}^{\infty}(k+n_a)\left(x_{k+n_a}^{(a)}-\frac{\delta_{aj}}{(k+n_j)z^{k+n_j}}\right)\frac{\partial}{\partial x_k^{(a)}}\right\}\tilde{\tau}_{\alpha+\delta_i-\delta_j}=0. \tag{5.2.3}$$

So, in a similar way as in [D], one deduces from (5.2.2) and (5.2.3) that

$$\tilde{\tau}_{\alpha+\delta_i-\delta_j}\tau_\alpha^{-2}L_{-1}\tau_\alpha-\tau_\alpha^{-1}L_{-1}\tilde{\tau}_{\alpha+\delta_i-\delta_j}=0.$$

Hence, we find that for all $\alpha\in Q$ and $1\le i,j\le s$:

$$\frac{1}{n_j}\left\{\frac{1}{\tau_\alpha}\left(\sum_{k=1}^{\infty}\frac{\partial}{\partial x_k^{(j)}}z^{-k-n_j}+\sum_{k=1}^{n_j}kx_k^{(j)}z^{k-n_j}\right.\right.$$
$$\left.+(\delta_{ij}-1+(\delta_j|\alpha)+\tfrac{1}{2}-\tfrac{1}{2}n_a)z^{-n_j}-n_jx_{n_i}^{(i)}\right)\tilde{\tau}_{\alpha+\delta_i-\delta_j}$$
$$\left.-\sum_{a=1}^{s}\frac{n_j}{n_a}\sum_{k=1}^{\infty}(k+n_a)x_{k+n_a}^{(a)}\frac{\partial}{\partial x_k^{(a)}}\left(\frac{\tilde{\tau}_{\alpha+\delta_i-\delta_j}}{\tau_\alpha}\right)\right\}=0. \tag{5.2.4}$$

Comparing this with (5.1.3), one finds

$$S\left(\left(\sum_{a=1}^{s}\left\{\left(\frac{1}{n_a}M(\alpha)L(\alpha)^{1-n_a}C^{(a)}(\alpha)\right)_- \right.\right.\right.$$
$$\left.\left.\left. - \frac{n_a-1}{2n_a}L(\alpha)^{-n_a}C^{(a)}(\alpha)\right\}P(\alpha)R(\alpha)\right)_{ij}\right) = 0 .$$

We thus conclude that the string equation induces for all $\alpha \in Q$:

$$\sum_{a=1}^{s}\left\{\left(\frac{1}{n_a}M(\alpha)L(\alpha)^{1-n_a}C^{(a)}(\alpha)\right)_- - \frac{n_a-1}{2n_a}L(\alpha)^{-n_a}C^{(a)}(\alpha)\right\} = 0. \qquad (5.2.5)$$

So, if (5.2.5) holds

$$N(\alpha) := \sum_{a=1}^{s}\left\{\frac{1}{n_a}M(\alpha)L(\alpha)^{1-n_a}C^{(a)}(\alpha) - \frac{n_a-1}{2n_a}L(\alpha)^{-n_a}C^{(a)}(\alpha)\right\}$$

is a differential operator that satisfies

$$\left[\sum_{a=1}^{s}L(\alpha)^{n_a}C^{(a)}(\alpha), N(\alpha)\right] = 1.$$

6. $W_{1+\infty}$ constraints

6.1. Let e_i, $1 \le i \le s$ be a basis of $\mathbb{C}^s$. In a similar way as in Section 2, we identify $(\mathbb{C}[t,t^{-1}])^s$ with $\mathbb{C}^\infty$, viz., we put

$$v^{(a)}_{-k-1/2} = t^k e_a. \qquad (6.1.1)$$

We can associate to $(\mathbb{C}[t,t^{-1}])^s$ s-copies of the Lie algebra of differential operators on $\mathbb{C}^\times$; it has as basis the operators (see [Ra] or [KRa]):

$$-t^{k+\ell}(\partial/\partial t)^\ell e_{ii}, \quad \text{for } k \in \mathbb{Z},\ \ell \in \mathbb{Z}_+,\ 1 \le i \le s.$$

We will denote this Lie algebra by D^s. Via (6.1.1) we can embed this algebra into $\overline{gl_\infty}$ and also into a_∞; one finds

$$-t^{k+\ell}(\partial/\partial t)^\ell e_{ii} \mapsto \sum_{m\in\mathbb{Z}} -m(m-1)\cdots(m-\ell+1)E^{(ii)}_{-m-k-1/2,-m-1/2}. \qquad (6.1.2)$$

It is straightforward, but rather tedious, to calculate the corresponding 2-cocycle; the result is as follows (see also [Ra] or [KRa]). Let $f(t), g(t) \in \mathbb{C}[t,t^{-1}]$; then

$$\mu(f(t)(\partial/\partial t)^\ell e_{aa}, g(t)(\partial/\partial t)^m e_{bb})$$
$$= \delta_{ab}\frac{\ell!m!}{(\ell+m+1)!}\mathrm{Res}_{t=0}\, dt\, f^{(m+1)}(t)g^{(\ell)}(t).$$

Hence in this way we get a central extension $\hat{D}^s = D^s \oplus \mathbb{C}c$ of D^s with Lie bracket

$$[f(t)(\partial/\partial t)^\ell e_{aa}+\alpha c, g(t)(\partial/\partial t)^m e_{bb}+\beta c]$$
$$=\delta_{ab}\Big\{(f(t)(\partial/\partial t)^\ell g(t)(\partial/\partial t)^m - g(t)(\partial/\partial t)^m f(t)(\partial/\partial t)^\ell)e_{aa}$$
$$+\frac{\ell!m!}{(\ell+m+1)!}\mathrm{Res}_{t=0}\, dt\, f^{(m+1)}(t)g^{(\ell)}(t)c\Big\}. \tag{6.1.3}$$

Since we have the representation $\hat{r}$ of a_∞, we find that

$$\hat{r}(-t^{k+\ell}(\partial/\partial t)^\ell e_{aa}) = \sum_{m\in\mathbb{Z}} m(m-1)\cdots(m-\ell+1) :\psi^{+(a)}_{-m-1/2}\psi^{-(a)}_{m+k+1/2}:.$$

In terms of the fermionic fields (2.1.2), we find

$$\sum_{k\in\mathbb{Z}}\hat{r}(-t^{k+\ell}(\partial/\partial t)^\ell e_{aa})z^{-k-\ell-1} = :\frac{\partial^\ell\psi^{+(a)}(z)}{z^\ell}\psi^{-(a)}(z):. \tag{6.1.4}$$

6.2. We will now express $-t^{k+\ell}(\partial/\partial t)^\ell e_{aa}$ in terms of the oscillators $\alpha_k^{(a)}$. For this purpose, we first calculate

$$:(y-z)\psi^{+(a)}(y)\psi^{-(a)}(z): = (y-z)\psi^{+(a)}(y)\psi^{-(a)}(z) - 1$$
$$= X_a(y,z) - 1,$$

where

$$X_a(y,z) = \left(\frac{y}{z}\right)^{\alpha_0^{(a)}} \exp\left(-\sum_{k<0}\frac{\alpha_k^{(a)}}{k}(y^{-k}-z^{-k})\right)$$
$$\times \exp\left(-\sum_{k>0}\frac{\alpha_k^{(a)}}{k}(y^{-k}-z^{-k})\right). \tag{6.2.1}$$

Then

$$:\frac{\partial^\ell\psi^{+(a)}(z)}{\partial z^\ell}\psi^{-(a)}(z): = \frac{1}{\ell+1}\left.\frac{\partial^{\ell+1}X_a(y,z)}{\partial y^{\ell+1}}\right|_{y=z}. \tag{6.2.2}$$

Notice that the right-hand-side of this formula is some normal ordered expression in the $\alpha_k^{(a)}$'s. For some explicit formulas of (6.2.2), we refer to the appendix of [AV].

6.3. In the rest of this section, we will show that $\hat{D}^s$ has a subalgebra that will provide the extra constraints, the so called W-algebra constraints on τ.

From now on we assume that τ is a τ-function of the $[n_1, n_2, \dots, n_s]$th reduced KP hierarchy, which satisfies the string equation. So, we assume that (4.2.3) and (5.2.1) holds. Hence, for all $\alpha \in \operatorname{supp}\tau$ both

$$Q(\alpha) := \sum_{a=1}^{s} L(\alpha)^{n_a} C^{(a)}(\alpha)$$

and

$$N(\alpha) = \sum_{a=1}^{s} \left\{ \frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} C^{(a)}(\alpha) - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha) \right\}$$

are differential operators. Thus, also $N(\alpha)^p Q(\alpha)^q$ is a differential operator, i.e.,

$$\left(\left(\sum_{a=1}^{s} \frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} \right)^p L(\alpha)^{qn_a} C^{(a)}(\alpha) \right)_- = 0$$

$$\text{for } p, q \in \mathbb{Z}_+. \tag{6.3.1}$$

Using (6.3.1),we are able to prove the following

Lemma 6.1. *For all $\alpha \in Q$ and $p, q \in \mathbb{Z}_+$*

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} z^{qn_a} \left(\frac{1}{n_a} z^{(1-n_a)/2} \frac{\partial}{\partial z} z^{(1-n_a)/2} \right)^p (V^+(\alpha, x, z))$$
$$\times E_{aa}{}^t V^-(\alpha, x', z) = 0. \tag{6.3.2}$$

Proof. Using Taylor's formula we rewrite the right-hand side of (6.3.2):

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} z^{qn_a} \left(\frac{1}{n_a} z^{(1-n_a)/2} \frac{\partial}{\partial z} z^{(1-n_a)/2} \right)^p (V^+(\alpha, x, z)) E_{aa}$$
$$\times \exp\left(\sum_{\ell=1}^{s} \sum_{k=1}^{\infty} (x_k^{(\ell)\prime} - x_k^{(\ell)}) \frac{\partial}{\partial x_k^{(\ell)}} \right) {}^t V^-(\alpha, x, z). \tag{6.3.3}$$

Since

$$\frac{\partial V^-(\alpha, x, z)}{\partial x_k^{(\ell)}} = -(P^-(\alpha, x, z) E_{\ell\ell} \partial^k P^-(\alpha, x, z)^{-1})_+ V^-(\alpha, x, z),$$

it suffices to prove that for all $m \geq 0$

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} z^{qn_a} \left(\frac{1}{n_a} z^{(1-n_a)/2} \frac{\partial}{\partial z} z^{(1-n_a)/2} \right)^p (V^+(\alpha, x, z))$$
$$\times E_{aa} \partial^{m t} V^-(\alpha, x, z) = 0. \tag{6.3.4}$$

Now, let

$$\sum_{a=1}^{s} z^{qn_a} \left(\frac{1}{n_a} z^{(1-n_a)/2} \frac{\partial}{\partial z} z^{(1-n_a)/2} \right)^p (V^+(\alpha, x, z)) E_{aa} = \sum_i S_i \partial^{-i} e^{x \cdot z}$$

and $V^-(\alpha, x, z) = \sum_j T_j \partial^{-j} e^{-x \cdot z}$, then (6.3.4) is equivalent to

$$
\begin{aligned}
0 &= \mathrm{Res}_{z=0}\, dz \sum_{i,j} S_i z^{-i} e^{x\cdot z} \partial^m (e^{-x\cdot z\, t}T_j(-z)^j) \\
&= \mathrm{Res}_{z=0}\, dz \sum_{i,j} \sum_{\ell=0}^{m} (-1)^{m-\ell+j} \binom{m}{\ell} S_i \partial^\ell ({}^tT_j) z^{m-i-j-\ell} \\
&= \sum_{\substack{0\le \ell \le m \\ i+j+\ell=m+1}} (-1)^{\ell+j} \binom{m}{\ell} S_i \partial^\ell ({}^tT_j).
\end{aligned}
\tag{6.3.5}
$$

On the other hand (6.3.1) implies that

$$
\begin{aligned}
0 &= \sum_i S_i \partial^{-i} \sum_j (-\partial)^{-j\, t}T_j)_- \\
&= \sum_{\substack{i,j \\ \ell \ge 0}} (-1)^j \binom{-i-j}{\ell} S_i \partial^\ell ({}^tT_j) \partial^{-i-j-\ell})_-.
\end{aligned}
$$

Now let $i+j+\ell = m+1$; then we obtain that for every $m \ge 0$

$$
\begin{aligned}
0 &= \sum_{\substack{0\le \ell \\ i+j+\ell=m+1}} (-1)^j \binom{-i-j}{\ell} S_i \partial^\ell ({}^tT_j) \\
&= \sum_{\substack{0\le \ell \le m \\ i+j+\ell=m+1}} (-1)^{\ell+j} \binom{m}{\ell} S_i \partial^\ell ({}^tT_j),
\end{aligned}
$$

which proves (6.3.5) □

Taking the (i,j)th coefficient of (6.3.3) one obtains

Corollary 6.2. *For all $\alpha \in Q$, $1 \le i, j \le s$ and $p, q \in \mathbb{Z}_+$ one has*

$$
\begin{aligned}
&\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} z^{qn_a} \left(\frac{1}{n_a} z^{(1-n_a)/2} \frac{\partial}{\partial z} z^{(1-n_a)/2} \right)^p (\psi^{+(a)}(z)) \\
&\times \tau_{\alpha+\delta_i-\delta_a} \otimes \psi^{-(a)}(z) \tau_{\alpha+\delta_a-\delta_j} = 0.
\end{aligned}
\tag{6.3.6}
$$

Notice that (6.3.6) can be rewritten as infinitely many generating series of Hirota bilinear equations (for the case $p = q = 0$ see [KV]).

6.4. The following lemma gives a generalization of an identity of Date, Jimbo, Kashiwara and Miwa [DJKM3] (see also [G]):

Lemma 6.3. *Let $X_b(y, w)$ be given by (6.2.1); then*

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \psi^{+(a)}(z) X_b(y,w) \tau_{\alpha+\delta_i-\delta_a} e^{\alpha+\delta_i-\delta_a} \otimes \psi^{-(a)}(z) \tau_{\alpha+\delta_a-\delta_j} e^{\alpha+\delta_a-\delta_j}$$
$$= (w-y)\psi^{+(b)}(y) \tau_{\alpha+\delta_i-\delta_b} e^{\alpha+\delta_i-\delta_b} \otimes \psi^{-(b)}(w) \tau_{\alpha+\delta_b-\delta_j} e^{\alpha+\delta_b-\delta_j}. \quad (6.4.1)$$

Proof. The left-hand-side of (6.4.1) is equal to

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \epsilon(\delta_a, \delta_i+\delta_j) z^{(\delta_a|\delta_i+\delta_j)-2} y^{(\delta_b|\alpha+\delta_i-\delta_a)} w^{-(\delta_b|\alpha+\delta_i-\delta_a)} \left(\frac{z-y}{z-w}\right)^{\delta_{ab}}$$
$$\times\, e^{x^{(b)}\cdot y - x^{(b)}\cdot w + x^{(a)}\cdot z} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k}(y^{-k}-w^{-k})\frac{\partial}{\partial x_k^{(b)}} + \frac{1}{k} z^{-k}\frac{\partial}{\partial x_k^{(a)}}\right) \tau_{\alpha+\delta_i-\delta_a} e^{\alpha+\delta_i}$$
$$\otimes\, e^{-x^{(a)}\cdot z} \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} z^{-k}\frac{\partial}{\partial x_k^{(a)}}\right) \tau_{\alpha+\delta_a-\delta_j} e^{\alpha-\delta_j}.$$

Recall the bilinear identity for $\beta = \alpha$:

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \psi^{+(a)}(z) \tau_{\alpha+\delta_i-\delta_a} e^{\alpha+\delta_i-\delta_a} \otimes \psi^{-(a)}(z) \tau_{\alpha+\delta_a-\delta_j} e^{\alpha+\delta_a-\delta_j} = 0.$$

Let $X_b(y,w) \otimes 1$ act on this identity; then

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \epsilon(\delta_a, \delta_i+\delta_j) z^{(\delta_a|\delta_i+\delta_j)-2} y^{(\delta_b|\alpha+\delta_i)} w^{-(\delta_b|\alpha+\delta_i)}$$
$$\times \left(\frac{y-z}{w-z}\right)^{\delta_{ab}} e^{x^{(b)}\cdot y - x^{(b)}\cdot w + x^{(a)}\cdot z}$$
$$\times \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k}(y^{-k}-w^{-k})\frac{\partial}{\partial x_k^{(b)}} + \frac{1}{k} z^{-k}\frac{\partial}{\partial x_k^{(a)}}\right) \tau_{\alpha+\delta_i-\delta_a} e^{\alpha+\delta_i}$$
$$\otimes\, e^{-x^{(a)}\cdot z} \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} z^{-k}\frac{\partial}{\partial x_k^{(a)}}\right) \tau_{\alpha+\delta_a-\delta_j} e^{\alpha-\delta_j} = 0.$$

Now, using this and the fact that

$$\frac{y-z}{w-z} = (\frac{y}{w})\frac{1-z/y}{1-z/w} = \frac{y-w}{z}\delta(w/z) + \frac{1-y/z}{1-w/z},$$

where $\delta(w/z) = \sum_{k\in\mathbb{Z}}(w/z)^k$ (such that $f(w,z)\delta(w/z) = f(w,w)\delta(w/z)$), we obtain that the left-hand side of (6.4.1) is equal to

$$(w-y)\; y^{(\delta_b|\alpha+\delta_i-\delta_b)} e^{x^{(b)}\cdot y} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} y^{-k}\frac{\partial}{\partial x_k^{(b)}}\right) \tau_{\alpha+\delta_i-\delta_b} e^{\alpha+\delta_i}$$

$$\otimes\, w^{-(\delta_b|\alpha+\delta_b-\delta_j)}e^{-x^{(b)}\cdot w}$$
$$\times \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}w^{-k}\frac{\partial}{\partial x_k^{(b)}}\right)\tau_{\alpha+\delta_b-\delta_j}e^{\alpha-\delta_j},$$

which is equal to the right-hand side of (6.4.1) □

Define $c_b(\ell,p)$ as follows:

$$\left(z^{(1-n_b)/2}\frac{\partial}{\partial z}z^{(1-n_b)/2}\right)^p=\sum_{\ell=0}^{p}c_b(\ell,p)z^{-n_bp+\ell}\left(\frac{\partial}{\partial z}\right)^{\ell}. \tag{6.4.2}$$

One also has

$$\left(M(\alpha)L(\alpha)^{-n_b+1}-\tfrac{1}{2}(n_b-1)L(\alpha)^{-n_b}\right)^p=\sum_{\ell=0}^{p}c_b(\ell,p)M(\alpha)^{\ell}L(\alpha)^{-n_bp+\ell}. \tag{6.4.3}$$

Then it is straightforward to show that

$$c_b(\ell,p)=\sum_{0\le q_0<q_1\cdots<q_{p-\ell-1}\le p-1}[(q_0+\tfrac{1}{2})(1-n_b)] \tag{6.4.4}$$
$$\times[(q_1+\tfrac{1}{2})(1-n_b)-1]\cdots[(q_{p-\ell-1}+\tfrac{1}{2})(1-n_b)-(p-\ell-1)].$$

Now using (6.4.2) and removing the tensor product symbol in (6.3.6), where we write x and x' for the first and the second component, respectively, of the tensor product, one gets:

$$\mathrm{Res}_{z=0}\, dz\sum_{a=1}^{s}\left(\frac{1}{n_a}\right)^p z^{qn_a}\sum_{\ell=0}^{p}c_a(\ell,p)z^{-n_ap+\ell}\frac{\partial^{\ell}\psi^{+(a)}(z)}{\partial z^{\ell}}$$
$$\times\,\tau_{\alpha+\delta_i-\delta_a}(x)\psi^{-(a)}(z)'\tau_{\alpha+\delta_a-\delta_j}(x')=0.$$

Using Lemma 6.3, this is equivalent to

$$\mathrm{Res}_{\substack{y=0\\z=0}}\, dy\,dz\sum_{a=1}^{s}\psi^{+(a)}(y)\sum_{b=1}^{s}\left(\frac{1}{n_b}\right)^p z^{qn_b}$$
$$\times\sum_{\ell=0}^{p}\frac{c_b(\ell,p)}{\ell+1}z^{-n_bp+\ell}\left.\frac{\partial^{\ell+1}X_b(w,z)}{\partial z^{\ell+1}}\right|_{w=z}$$
$$\times\,\tau_{\alpha+\delta_i-\delta_a}(x)e^{\alpha+\delta_i-\delta_a}\psi^{-(a)}(y)'\tau_{\alpha+\delta_a-\delta_j}(x')(e^{\alpha+\delta_a-\delta_j})'=0.$$

Now, recall (6.1.4) and (6.2.2), then

$$
\begin{aligned}
&\mathrm{Res}_{z=0}\, dz \sum_{b=1}^{s} \left(\frac{1}{n_b}\right)^p z^{qn_b} \sum_{\ell=0}^{p} \frac{c_b(\ell,p)}{\ell+1} z^{-n_b p+\ell} \left.\frac{\partial^{\ell+1} X_b(w,z)}{\partial w^{\ell+1}}\right|_{w=z} \\
&= \mathrm{Res}_{z=0}\, dz \sum_{b=1}^{s} \left(\frac{1}{n_b}\right)^p z^{qn_b} \sum_{\ell=0}^{p} c_b(\ell,p) z^{-n_b p+\ell} : \frac{\partial^\ell \psi^{+(b)}(z)}{\partial z^\ell} \psi^{-(b)}(z) : \\
&= \sum_{b=1}^{s} \left(\frac{1}{n_b}\right)^p \sum_{\ell=0}^{p} c_b(\ell,p) \hat{r} \left(-t^{(q-p)n_b+\ell} \left(\frac{\partial}{\partial t}\right)^\ell e_{bb} \right) \\
&= -\sum_{b=1}^{s} \left(\frac{1}{n_b}\right)^p \hat{r} \left(t^{n_b q} \left(t^{(1-n_b)/2} \frac{\partial}{\partial t} t^{(1-n_b)/2} \right)^p e_{bb} \right) \\
&= -\sum_{b=1}^{s} \hat{r} \left(\left(t^{(n_b-1)/2} t^{n_b q} \left(\frac{\partial}{\partial t^{n_b}}\right)^p t^{(1-n_b)/2} \right) e_{bb} \right) \\
&= \sum_{b=1}^{s} \hat{r} \left(t^{(n_b-1)/2} \left(-\lambda_b^q \left(\frac{\partial}{\partial \lambda_b}\right)^p \right) t^{(1-n_b)/2} \right) e_{bb}) \quad \text{where } \lambda_b = t^{n_b} \\
&\stackrel{\text{def}}{=} W_{q-p}^{(p+1)}.
\end{aligned} \tag{6.4.5}
$$

Hence, (6.3.6) is equivalent to

$$
\begin{aligned}
&\mathrm{Res}_{y=0}\, dy \sum_{a=1}^{s} \psi^{+(a)}(y) W_{q-p}^{(p+1)} \\
&\quad \times \tau_{\alpha+\delta_i-\delta_a}(x) e^{\alpha+\delta_i-\delta_a} \psi^{-(a)}(y)' \tau_{\alpha+\delta_a-\delta_j}(x') (e^{\alpha+\delta_a-\delta_j})' = 0.
\end{aligned} \tag{6.4.6}
$$

If we ignore the cocycle term for a moment, then it is obvious from the sixth line of (6.4.5), that the elements $W_q^{(p+1)}$ are the generators of the W-algebra $W_{1+\infty}$ (the cocycle term, however, will be slightly different). Up to some modification of the elements $W_0^{(p+1)}$, one gets the standard commutation relations of $W_{1+\infty}$, where $c = nI$.

As the next step, we take in (6.4.6) $x_k^{(i)} = x_k^{(i)\prime}$, for all $k \in \mathbb{N}$, $1 \le i \le s$; we then obtain

$$
\begin{aligned}
&\frac{\partial}{\partial x_1^{(i)}} \left(\frac{W_{q-p}^{(p+1)} \tau_\alpha}{\tau_\alpha} \right) = 0 && \text{if } i = j, \\
&\tau_{\alpha+\delta_i-\delta_j} W_{q-p}^{(p+1)} \tau_\alpha = \tau_\alpha W_{q-p}^{(p+1)} \tau_{\alpha+\delta_i-\delta_j} && \text{if } i \ne j.
\end{aligned} \tag{6.4.7}
$$

The last equation means that for all $\alpha, \beta \in \operatorname{supp} \tau$ one has

$$
\frac{W_{q-p}^{(p+1)} \tau_\alpha}{\tau_\alpha} = \frac{W_{q-p}^{(p+1)} \tau_\beta}{\tau_\beta}. \tag{6.4.8}
$$

Next we divide (6.4.6) by $\tau_\alpha(x)\tau_\alpha(x')$, of course only for $\alpha \in \operatorname{supp} \tau$, and use (6.4.8). Then for all $\alpha, \beta \in \operatorname{supp} \tau$ and $p, q \in \mathbb{Z}_+$ one has

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}}\right) \left(\frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)}\right)$$
$$\times \frac{\psi^{+(a)}(z)\tau_{\alpha+\delta_i-\delta_a}(x)}{\tau_\alpha(x)} e^{\alpha+\delta_i-\delta_a} \frac{\psi^{-(a)}(z)'\tau_{\alpha+\delta_a-\delta_j}(x')}{\tau_\alpha(x')} (e^{\alpha+\delta_a-\delta_j})' = 0.$$

Since one also has the bilinear identity (3.3.3) (see also (2.4.1), (2.4.2)), we can subtract that part and thus obtain the following

Lemma 6.4. *For all $\alpha, \beta \in supp\ \tau$ and $p, q \in \mathbb{Z}_+$ one has*

$$\mathrm{Res}_{z=0}\, dz \sum_{a=1}^{s} \left\{\exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}}\right) - 1\right\} \left(\frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)}\right)$$
$$\times \frac{\psi^{+(a)}(z)\tau_{\alpha+\delta_i-\delta_a}(x)}{\tau_\alpha(x)} e^{\alpha+\delta_i-\delta_a} \frac{\psi^{-(a)}(z)'\tau_{\alpha+\delta_a-\delta_j}(x')}{\tau_\alpha(x')} (e^{\alpha+\delta_a-\delta_j})' = 0. \quad (6.4.9)$$

Define

$$S(\beta, p, q, x, z) := \sum_{a=1}^{s} \left\{\exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}}\right) - 1\right\} \left(\frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)}\right) E_{aa}.$$

Notice that the first equation of (6.4.7) implies that $\partial \circ S(\beta, p, q, x, \partial) = S(\beta, p, q, x, \partial) \circ \partial$. Then viewing (6.4.9) as the (i, j)th entry of a matrix, (6.4.9) is equivalent to

$$\mathrm{Res}_{z=0}\, dz\, P^+(\alpha)R^+(\alpha)S(\beta, p, q, x, \partial)e^{x\cdot z}\ {}^t(P^-(\alpha)'R^-(\alpha)'e^{-x'\cdot z}) = 0. \quad (6.4.10)$$

Now using Lemma 3.1, one deduces

$$(P^+(\alpha)R^+(\alpha)S(\beta, p, q, x, \partial)R^+(\alpha)^{-1}P^+(\alpha)^{-1})_- = 0; \quad (6.4.11)$$

hence

$$P^+(\alpha)S(\beta, p, q, x, \partial)P^+(\alpha)^{-1} = (P^+(\alpha)S(\beta, p, q, x, \partial)P^+(\alpha)^{-1})_- = 0.$$

So $S(\beta, p, q, x, \partial) = 0$ and therefore

$$\left\{\exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial x_k^{(a)}}\right) - 1\right\} \left(\frac{W_{q-p}^{(p+1)} \tau_\beta(x)}{\tau_\beta(x)}\right) = 0,$$

from which we conclude that

$$W_k^{(p+1)}\tau_\beta = \text{constant}\ \tau_\beta \quad \text{for all}\ -k \le p \ge 0. \quad (6.4.12)$$

In order to determine the constants on the right-hand side of (6.4.12) we calculate the Lie brackets,

$$\left[W_{-1}^{(2)}, \frac{-1}{k+p+1} W_{k+1}^{(p+1)}\right] \tau_\beta = 0\,, \quad (6.4.13)$$

and thus obtain the main result

Theorem 6.5. *The following two conditions for $\tau \in F^{(0)}$ are equivalent:*

(1) τ is a τ-function of the $[n_1, n_2, \dots, n_s]$th reduced s-component KP hierarchy which satisfies the string Eq. (5.2.1).

(2) For all $p \geq 0, k \geq -p$:

$$(W_k^{(p+1)} + \delta_{k0} c_p)\tau = 0, \tag{6.4.14}$$

where

$$\begin{aligned} c_p &= \frac{1}{2p+2} \sum_{a=1}^{s} \left(\frac{-1}{n_a}\right)^{p+1} \sum_{\ell=0}^{p} \ell \cdot \ell! \binom{n_a+\ell}{\ell+2} \\ &\quad \times \sum_{0 \leq q_0 < q_1 \cdots < q_{p-\ell-1} \leq p-1} [(q_0 + \tfrac{1}{2})(n_a - 1)] \\ &\quad \times [(q_1 + \tfrac{1}{2})(n_a - 1) + 1] \cdots [(q_{p-\ell-1} + \tfrac{1}{2})(n_a - 1) + p - \ell - 1] \\ &= \frac{1}{1+p} \sum_{a=1}^{s} \sum_{j=1}^{n_a} \left(\frac{n_a - 2j + 1}{2n_a}\right) \left(\frac{n_a - 2j + 1}{2n_a} - 1\right) \cdots \left(\frac{n_a - 2j + 1}{2n_a} - p\right). \end{aligned} \tag{6.4.15}$$

For $p = 0, 1$, the constants c_p are equal to 0, respectively $\sum_{a=1}^{s} (n_a^2 - 1)/24 n_a$.

Proof of Theorem 6.5. The case (2) $\Rightarrow$ (1) is trivial. For the implication (1) $\Rightarrow$ (2), we only have to calculate the left-hand side of (6.4.13). It is obvious that this is equal to $(W_k^{(p+1)} + c_{p,k})\tau_\beta$, where

$$c_{p,k} = \mu\left(W_{-1}^{(2)}, \frac{-1}{k+p+1} W_{k+1}^{(p+1)}\right).$$

It is clear from (6.1.3) that $c_{p,k} = 0$ for $k \neq 0$. So from now on we assume that $k = 0$ and $c_p = c_{p,0}$. Then

$$\begin{aligned} c_p &= \frac{-1}{p+1} \mu(W_{-1}^{(2)}, W_1^{(p+1)}) \\ &= \frac{-1}{p+1} \sum_{a=1}^{s} (\frac{1}{n_a})^{p+1} \mu\left(\tfrac{1}{2}(1 - n_a) t^{-n_a} + t^{1-n_a} \frac{\partial}{\partial t}, \sum_{\ell=0}^{p} c_a(\ell, p) t^{n_a+\ell} \left(\frac{\partial}{\partial t}\right)^{\ell}\right) \\ &= \frac{1}{2p+2} \sum_{a=1}^{s} \left(\frac{1}{n_a}\right)^{p+1} \sum_{\ell=0}^{p} (-1)^{\ell+1} \ell \cdot \ell! \binom{n_a+\ell}{\ell+2} c_a(\ell, p), \end{aligned}$$

which equals (6.4.15).

It is possible to find a shorter expression for c_p, viz., if one writes

$$\begin{aligned} W_{q-p}^{(p+1)} &= \sum_{a=1}^{s} \left(\frac{1}{n_a}\right)^{p} t^{(q-p)n_a} (T + \tfrac{1}{2}(1 - n_a))(T + \tfrac{1}{2}(1 - 3n_a)) \cdots \\ &\quad \times (T + \tfrac{1}{2}(1 - (2p-1)n_a)) e_{aa}, \end{aligned}$$

where $T = t\partial/\partial t$, then using results from [KRa] one finds that

$$c_p = \frac{1}{1+p} \sum_{a=1}^{s} \sum_{j=1}^{n_a} \left(\frac{n_a - 2j + 1}{2n_a}\right) \left(\frac{n_a - 2j + 1}{2n_a} - 1\right) \cdots \left(\frac{n_a - 2j + 1}{2n_a} - p\right). \quad \square$$

7. A geometrical interpretation of the string equation on the Sato Grassmannian

7.1. It is well-known that every τ-function of the 1-component KP hierarchy corresponds to a point of the Sato Grassmannian Gr (see e.g. [S]). Let H be the space of formal Laurent series $\sum a_n t^n$ such that $a_n = 0$ for $n \gg 0$. The points of Gr are those linear subspaces $V \subset H$ for which the natural projection π_+ of V into $H_+ = \{\sum a_n t^n \in H \mid a_n = 0 \text{ for all } n < 0\}$ is a Fredholm operator. The big cell Gr^0 of Gr consists of those V for which π_+ is an isomorphism.

The connection between Gr and the semi-infinite wedge space is made as follows. Identify $v_{-k-1/2} = t^k$. Let V be a point of Gr and $w_0(t), w_{-1}(t), \ldots$ be a basis of V; then we associate to V the following element in the semi-infinite wedge space:

$$w_0(t) \wedge w_{-1}(t) \wedge w_{-2}(t) \wedge \cdots.$$

If τ is a τ-function of the nth KdV hierarchy, then τ corresponds to a point of Gr that satisfies $t^n V \subset V$ (see e.g. [SW, KS]).

In the case of the s-component KP hierarchy and its $[n_1, n_2, \ldots, n_s]$-reduction we find it convenient to represent the Sato Grassmannian in a slightly different way. Let now H be the space of formal Laurent series $\sum a_n t^n$ such that $a_n \in \mathbb{C}^s$ and $a_n = 0$ for $n \gg 0$. The points Gr are those linear subspaces $V \subset H$ for which the projection π_+ of V into $H_+ = \{\sum a_n t^n \in H \mid a_n = 0 \text{ for all } n < 0\}$ is a Fredholm operator. Again, the big cell Gr^0 of Gr consists of those V for which π_+ is an isomorphism. The connection with the semi-infinite wedge space is of course given in a similar way via (2.1.1):

$$v_{nj-N_a-p+1/2} = v^{(a)}_{n_a j-p+1/2} = t^{-n_a j+p-1} e_a,$$

where e_a, $1 \le a \le s$, is an orthonormal basis of $\mathbb{C}^s$.

It is obvious that τ-functions of the $[n_1, n_2, \ldots, n_s]$th reduced s-component KP hierarchy correspond to those subspaces V for which

$$\left(\sum_{a=1}^{s} t^{n_a} E_{aa}\right) V \subset V. \tag{7.1.1}$$

7.2. The proof that there exists a τ-function of the $[n_1, n_2, \ldots n_s]$th reduced KP hierarchy that satisfies the string equation is in great detail similar to the proof of Kac and Schwarz [KS] in the principal case, i.e., the nth KdV case.

Recall the string equation $L_{-1}\tau = H_{-1}\tau = 0$. Now modify the origin by replacing x_{n_a+1} by $x_{n_a+1} - 1$ for all $1 \le a \le s$. Then the string equation transforms to

$$\left(L_{-1}-\sum_{a=1}^{s}\frac{n_a+1}{n_a}\frac{\partial}{\partial x_1^{(a)}}\right)\tau=0,$$

or equivalently

$$\left(H_{-1}-\sum_{a=1}^{s}\frac{n_a+1}{n_a}\frac{\partial}{\partial x_1^{(a)}}\right)\tau=0.$$

In terms of elements of $\hat{D}$ this is

$$\hat{r}(-A)\tau=0, \tag{7.2.1}$$

where

$$A=\sum_{a=1}^{s}\frac{1}{n_a}\left((n_a+1)t+t^{1-n_a}\frac{\partial}{\partial t}-\tfrac{1}{2}(n_a-1)t^{-n_a}\right)E_{aa}. \tag{7.2.2}$$

Hence for $V\in Gr$, this corresponds to

$$AV\subset V. \tag{7.2.3}$$

Now we will prove that there exists a subspace V satisfying (7.1.1) and (7.2.3). We will first start by assuming that $m=n_1=n_2=\cdots=n_s$ (this is the case that $L(\alpha)^m$ is a differential operator). For this case we will show that there exists a unique point in the big cell Gr^0 that satisfies both (7.1.1) and (7.2.3). So assume that $V\in Gr^0$ and that V satisfies these two conditions. Since the projection π_+ on H_+ is an isomorphism, there exist $\phi_a\in V$, $1\le a\le s$, of the form $\phi_a=e_a+\sum_{i,a}c_{i,a}t^{-i}$, with $c_{i,a}=\sum_{b=1}^{s}c_{i,a}^{(b)}e_b\in\mathbb{C}^s$. Now $A^p\phi_a=t^pe_a+$lower degree terms; hence these functions for $p\ge 0$ and $1\le a\le s$ form a basis of V. Therefore, $t^m\phi_a$ is a linear combination of $A^p\phi_b$; it is easy to observe that $A^m\phi_a=$ constant $t^m\phi_a$. Using this we find a recurrent relation for the $c_{i,a}^{(b)}$'s:

$$\left(\frac{m+1}{m}\right)^{m-1}ic_{i,a}^{(b)}=\sum_{\ell=1}^{m-1}d_{m,i,\ell}c_{i-\ell(m+1),a}^{(b)}; \tag{7.2.4}$$

here the $d_{m,i,\ell}$ are coefficients depending on m,i,ℓ, which can be calculated explicitly using (7.2.2). Since $c_{0,a}^{(b)}=\delta_{ab}$ and $c_{i,a}^{(b)}=0$ for $i<0$ one deduces from (7.2.4) that $c_{i,a}^{(b)}=0$ if $b\ne a$, and $c_{i,a}^{(a)}=0$ if $i\ne(m+1)k$ with $k\in\mathbb{Z}$. So the ϕ_a for $1\le a\le s$ can be determined uniquely. More explicitly, all ϕ_a are of the form $\phi_a=\phi^{(m)}e_a$, with

$$\phi^{(m)}=\sum_{i=1}^{\infty}b_i^{(m)}t^{-(m+1)i}, \tag{7.2.5}$$

where the b_i do not depend on a and satisfy

$$\left(\frac{m+1}{m}\right)^{m-1}i(m+1)b_i^{(m)}=\sum_{\ell=1}^{m-1}d_{m,i,\ell}b_{i-\ell}^{(m)}.$$

Thus the space $V\in Gr^0$ is spanned by $t^{km}A^\ell\phi_a$ with $1\le a\le s$, $k\in\mathbb{Z}_+$, $0\le\ell<m$.

Notice that in the case that all $n_a = 1$ we find that $V = H_+$, meaning that the only solution of (7.1.1) and (7.2.3) in Gr^0 is τ = constant e^0, corresponding to the vacuum vector $|0\rangle$.

If not all n_a are the same, then it is obvious that there still is a $V \in Gr^0$ satisfying (7.1.1) and (7.2.3), viz., V spanned by $t^{kn_a}A^{\ell_a}\phi^{(n_a)}e_a$, with $1 \le a \le s$, $k \in \mathbb{Z}_+$, $0 \le \ell_a < n_a$, where $\phi^{(n_a)}$ is the unique solution determined by (7.2.5). However, at the present moment we do not know if this $V \in Gr^0$ is still unique in Gr^0.

Acknowledgements

I would like to thank Victor Kac for valuable discussions, and the Mathematical Institute of the University of Utrecht for the computer and e-mail facilities. I want to dedicate this paper to the memory of my father, Noud van de Leur, who died quite suddenly in the period I was writing this article.

References

[AV] M. Adler and P. van Moerbeke, A matrix integral solution to two-dimensional W_p-gravity, Comm. Math. Phys. 147 (1992) 25–56.

[BT] M. Bergvelt and A. ten Kroode, Partitions, vertex operator constructions and multi-component KP equations, preprint University of Illinois, to appear in Pacific Journal of Mathematics.

[DJKM1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Operator approach to the Kadomtsev–Petviashvili equation. Transformation groups for soliton equations. III, J. Phys. Soc. Japan 50 (1981) 3806–3812.

[DJKM2] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy, Publ. Res. Inst. Math. Sci. 18 (1982) 1077–1110.

[DJKM3] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, in: *Nonlinear Integrable Systems – Classical Theory and Quantum Theory*, eds. M. Jimbo and T. Miwa (World Scientific, 1983) pp. 39–120.

[D] L.A. Dickey, Additional symmetries of KP, Grassmannian, and the string equation II, preprint University of Oklahoma (1992).

[Dij] R. Dijkgraaf, Intersection theory, integrable hierarchies and topological field theory, preprint IASSNS-HEP-91, hep-th 9201003.

[FK] I.B. Frenkel and V.G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980) 23–66.

[FKN] M. Fukuma, H. Kawai and R. Nakayama, Infinite dimensional Grassmannian structure of two-dimensional quantum gravity, Comm. Math. Phys. 143 (1992) 371–403.

[G] J. Goeree, *W*-constraints in 2d quantum gravity, Nucl. Phys. B 358 (1991) 737–157.

[JM] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. Res. Inst. Math. Sci. 19 (1983) 943–1001.

[K] V.G. Kac, *Infinite Dimensional Lie Algebras*, Progress in Math., Vol. 44 (Birkhäuser, Boston, 1983; 2nd Ed., Cambridge Univ. Press, 1985; 3rd Ed. 1990).

[KP1] V.G. Kac and D.H. Peterson, 112 constructions of the basic representation of the loop group of E_8, in: *Proc. Symp. Anomalies, Geometry, Topology (Argonne)*, eds. W.A. Bardeen and A.R. White (World Scientific, 1985) pp. 276–298.

[KP2] V.G. Kac and D.H. Peterson, Lectures on the infinite wedge representation and the MKP hierarchy, Sem. Math. Sup. 102 (Presses Univ. Montreal, Montreal, 1986) 141–184.

[KP3] V.G. Kac and D.H. Peterson, Spin and wedge representations of infinite dimensional Lie algebras and groups, Proc. Nat. Acad. Sci U.S.A. (1981) 3308–3312.

[KR] V.G. Kac and A.K. Raina, *Bombay Lectures on Highest Weight Representations of Infinite-Dimensional Lie Algebras*, Advanced Ser. in Math. Phys., Vol. 2 (World Scientific, 1987).

[KRa] V. Kac and A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, Comm. Math. Phys. 157 (1993) 429–457.

[KS] V. Kac and A. Schwarz, Geometric interpretation of partition function of 2d gravity, Phys. Lett. B 257 (1991) 363.

[KV] V. Kac and J. van de Leur, The n-component KP hierarchy and representation theory, in: *Important Developments in Soliton Theory*, eds. A.S. Fokas and V.E. Zakharov, Springer Series in Nonlinear Dynamics (1993) pp. 302–343.

[L] J. Lepowsky, Calculus of twisted vertex operators, Proc. Nat. Acad. Sci. U.S.A. 82 (1985) 8295–8299.

[Ra] A.O. Radul, Lie algebras of differential operators, their central extensions, and W-algebras, Funct. Anal. Appl. 25 (1991) 33–49.

[S] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, Res. Inst. Math. Sci. Kokyuroku 439 (1981) 30–46.

[SW] G. Segal and G. Wilson, Loop groups and equations of KdV type, Inst. Hautes Etudes Sci. Publ. Math. 63 (1985) 1–64.

[TV] F. ten Kroode and J. van de Leur, Bosonic and fermionic realizations of the affine algebra $\widehat{gl}_n$, Comm. Math. Phys. 137 (1991) 67–107.

[UT] K. Ueno and K. Takasaki, Toda lattice hierarchy, in: Adv. Stud. Pure Math., Vol. 4 (North-Holland, 1984) pp. 1–95.